Stability of Linear Systems

Stability of systems in the time domain	**Stability using Characteristic equation & Routh – Hurwitz Criterion**	**Stability using Liapunov function**	Stability using Liapunov function	
				Objective type questions:

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Problem 3.1: Stability of systems in the time domain

Fig.1 shows a system with state-variable feedback,

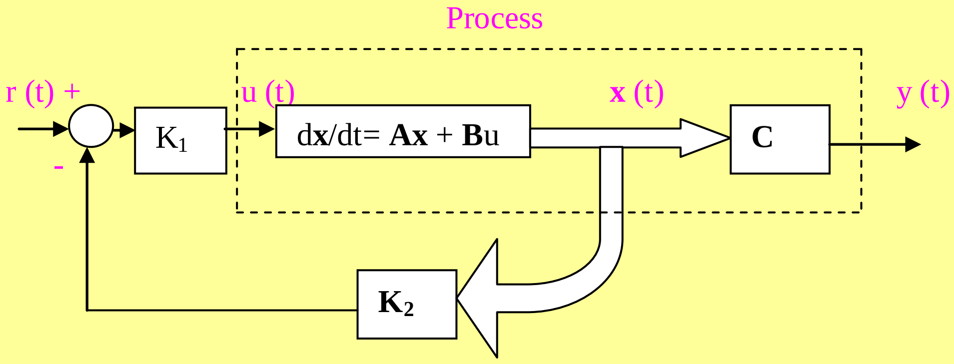

Fig.1

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -6 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}$$

(i) Determine the open loop transfer function, $Y(s)/U(s)$, and show that it represents an unstable process.

(ii) Show that the system with state variable feedback is represented by

$$d\mathbf{x}/dt = \mathbf{A_f}\,\mathbf{x} + K_1\,\mathbf{B}r$$

where $A_f = \mathbf{A} - \mathbf{B}K_1\,\mathbf{K_2}$.

Note that K_1 is a scalar quantity and $\mathbf{K_2}$ is a row vector.

(iii) Determine the gain K_1 and the state variable feedback row vector $\mathbf{K_2}$ such that the desired closed loop transfer function, which makes the overall system stable, is given by

$$Y(s)/R(s) = 40(s+3)/\,[s^3+12s^2+62s+120].$$

Solution:

(i)

$Y(s)/u(s) = (s+3)/\,[s^3+0s^2+6s+30] = (s+3)/(s+2)\,(s^2-2s+10)$

Use Routh-Hurwitz criterion to show that the system is unstable

(ii)

It can be shown that $\mathbf{A_f} = \mathbf{A} - \mathbf{B}K_1\mathbf{K_2}$

(iii)

From the denominator of the closed loop transfer function,

$$\mathbf{A_f} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -120 & -62 & -12 \end{pmatrix}$$

$$\mathbf{B}K_{1\,K2} = \mathbf{A}-\mathbf{A_f} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 100 & 56 & 12 \end{pmatrix}$$

A comparison of the numerator gives $K_1 = 40$

Hence $\mathbf{K_2} = \begin{bmatrix} 2.5 & 1.4 & 0.3 \end{bmatrix}$

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Problem 3.2: Stability using Characteristic equation & Routh – Hurwitz Criterion

The spread of an epidemic disease in a population is represented by the following equations:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

The population is made up of three groups as follows:

x_1 = group of people susceptible to the epidemic disease
x_2 = group infected with the disease
x_3 = group removed from x_1 due to immunization, death or isolation from x_1
x_3 is dependent on x_1 and x_2, and does not affect x_1 and x_2.

$u_1(t)$ = rate at which new susceptibles are added to the population
$u_2(t)$ = rate at which new infected people are added to the population.

For a closed population, $u_1(t) = u_2(t) = 0$.

The equilibrium point for this system, $x_1 = x_2 = 0$, is the point to which the system settles to the rest condition and is obtained by setting $d\mathbf{x}/dt = \mathbf{0}$.

Determine the conditions of stability of this system for the epidemic disease to be eliminated from the population.

Hint: Use the characteristic equation of the system and the Routh -Hurwitz criterion

Solution:

The characteristic eqn is obtained from:

$\det(sI - A) = 0$

$s^3 + (\alpha+\gamma)\, s2 + (\alpha\gamma+\beta^2)\, s = 0$

Routh array

s^3	1	$+(\alpha\gamma+\beta^2)$	
s^2	$(\alpha+\gamma)$	0	
s	$(\alpha\gamma+\beta^2)$	0	
s^0	0		

System is stable when $(\alpha+\gamma) > 0$ and $(\alpha\gamma+\beta^2) > 0$

If these conditions are satisfied, the epidemic is eliminated from the population.

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Problem 3.3: Stability using Liapunov function

(a) Summarize the concepts of stability for the following situations:

- i At least one of the system poles is in the right-half s=plane
- ii At least one of the system poles are repeated on the imaginary axis
- iii All poles on the imaginary axis are simple and none of these poles are present in the input
- iv All the system poles are inside the left-half of the s-plane
- v At least one of the system poles on the imaginary axis is present in the form of a pole in the input.

(b) The differential equation of a second- order system is

$$d^2e/dt^2 + k\, de/dt + e = 0, \ k>0.$$

Using the state variables $x_1 = e$ and $x^2 = de/dt$, examine the stability of the system by choosing the Liapunov function $V=x_1^2+x_2^2$.

Solution:

(a)

i unstable
ii unstable

iii non-asymptotically stable

iv asymptotically stable
v instability due to resonance

(b) $dx1/dt = x2$

$dx2/dt = -kx2-x1$

$V = x1^2 + x2^2$

$dV/dt = 2x1dx1/dt + 2x2dx2/dt = 2x1x2 + 2x2(-kx2-x1) = -2kx2^2$. This is negative definite. Hence,

the system is stable.

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Problem 3.4: Stability using Liapunov function

(a) What is the main theme and the main advantage of the Liapunov's method osf stability analysis.

(b) The defining equations of a second-order autonomous system are:

$$dx_1/dt = x_2$$
$$dx_2/dt = - K_1 x_1 - K_2 x_2.$$

Investigate the stability of the system by Liapunov's method using the scalar function:

$V = x_1^2 + x_2^2$

Solution:

(b) v is + definite
V (0) =0
V(x) is greater than zero for all x not=0
W= dV/dt = (delV/delx1) dx1/dt +delV/delx2.dx2/dt
= 2x1dx1/dt +zx2dx2/dt
= 2x1x2 +2x2(-k2x2-k1x1)
= [2x1(1-k1)-2k2x2] x2
For k1=1 and k2 greater than o, W= dV/dt $=-2k2x2^2$

Therefore for any value of x1, W=0 when x2=0
I.e., W is negative semi-definite
System stability is guaranteed for k1 =1 and k2 greater than zero.

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Objective type questions:

(i) A system with the following characteristic equation

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

is

(a) stable
(b) unstable with two roots lying in the right half of the s-plane
(c) unstable with one root lying in the right half of the s-plane

Ans: (b)

(ii) Which of the following systems, whose characteristic equations are given below, is stable?

(a) $s^2+2s+3=0$
(b) $s^4+2s^2+3s+4=0$
(c) $s^2-2s+3=0$

Ans: (a)

(iii) A system is called absolutely stable if any oscillations set up in the system are

(a) self-sustaining and tend to last indefinitely
(b) eventually damped out
(c) not enough to change the parameters of the system

Ans: (b)

(iv) For the Routh -Hurwitz array:

S^3	1	100
S^2	4	500
S^1	-25	0
S^0	500	0

the number of roots in the right-half s-plane is equal to
(a) one
(b) two
(c) three
(d)four

Ans: (a)

(v) Which one of the following systems whose characteristic equations are given below is stable?

a. $f(s) = s^5+s^4 +3s^3+9s^2+16 s + 10$
b. $f(s) = s^6+3s^5+5s^4 9s^3+8s^2+6 s + 4$
c. $f(s) = s^4 +2s^3+3s^2+6 s + K$
d. $f(s) =s^6 + s^5+3s^4 +2s^3+11s^2+9 s + 9$

Ans:a

(vi) Which one of the following statements is not true?

(a) Routh criterion tells us whether or not there are positive roots in a polynomial equation without actually solving it.
(b) Routh criterion applies to polynomials with only a finite number of terms.
(c) Routh criterion gives information about absolute stability as well as relative stability.
(d) Routh criterion gives information only about the absolute stability.

Ans: (c)

(vii) Which one of the systems having the following characteristic equations is stable?
(a) $s^3 +s^2+2s+24=0$
(b) $s^3+2s^2+4s+9=0$
(c) $s^4+5s^3+20s^2+40s+50=0$
(d) $5s^6+8s^5+12s^4+20s^3+100s^2+150s+200=0$

Ans: ©

(viii) Which of the following systems is stable?

(a) $as^4 + bs^2 + cs + d = 0$
(b) $as^2 + bs + c = 0$
(c) $as^2 - bs + c = 0$
(d) $-as^2 + bs - c = 0$

Ans: (b)

(ix) The open loop transfer function of a system is

$$G(s) = \frac{K(s+z_1)(s+z_2)}{s^3}.$$

The root locus is shown in Fig.2

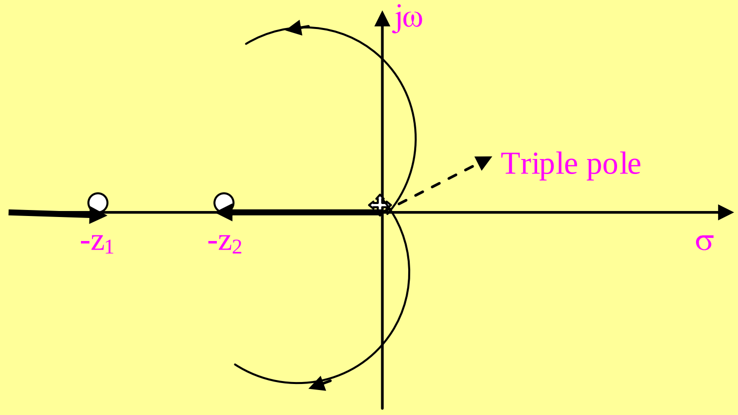

Fig.2

The system will be

(a) stable
(b) unstable
(c) conditionally stable ,becomes unstable at high gain
(d) conditionally stable, becomes unstable if gain is too low

Ans: (c)

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