NONLINEAR SYSTEMS

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Problem 10.1: Describing function of a nonlinear device (output is the cube of its input)

(a) A nonlinear electronic device produces an output that is the cube of its input (i.e. $y = x^3$). Derive the describing function of the device.

(b) The input x (t) and the output y (t) of a nonlinear system are related through the nonlinear differential equation

$$y(t) = x^2(dx/dt) + 2x.$$

Determine the describing function of the system.

Solution:

(a)

Let $x = A \cos wt$. Then $y = A^3 \cos^3 wt$.

$\text{Cos}3x = 4 \cos^3 x - 3\cos x$
$\text{Cos}^3 x = (3\cos x)/4 + (\cos 3x)/3$

Describing function = $y/x = \dfrac{A^3 [(3\cos wt/4) + (\cos 3wt/4)]}{A \cos wt}$

Discard the third harmonic. Then

Describing function $= 3A^2/4$

(b)

Since $x = A \sin wt$,
$Y = A^3 w \sin^2 wt \cos wt + 2A \sin wt$
$= A^3 w \cos wt - A^3 w \cos^3 wt + 2A \sin wt$
$= A^3 w \cos wt - (A^3 w/4)(\cos 3wt + 3\cos wt) + 2A \sin wt$
$= (A^3 w/4) \cos wt + 2A \sin wt - (A^3 w/4) \cos 3wt$
Describing function = $[(A^4 w^2/16) + 4]^{1/2} \exp [j\tan^{-1}(A^2 w/8)]$

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Problem 10.2: Stability of a system described by its state equation and assumed Liapunov function

Study the stability of the two systems described by their state-variable equations and assumed Liapunov functions.

System I: $dx_1/dt = x_2$
$dx_2/dt = -x_1 - x_2^3$

Assume $V = x_1^2 + x_2^2$

System II: $dx_1/dt = -x_1 + 2x_1^2 x_2$
$dx_2/dt = -x_2$

Assume $V = (1/2)x_1^2 + x_2^2$

Solution:

System I

dV/dt = 2 x1 dx1/dt +2x2dx2/dt=2x1x2+2x2(-x1-x23) =-2x34

dV/dt is less than zero for all nonzero values of x2. Hence the system is asymptotically stable.

System II

dV/dt= x1dx1/dt+2x2dx2/dt=-x12(1-2x1x2)-2x23

dV/dt is negative if 1-2x1x2 is greater than zero. This defines the region of stability in the state – space.

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Problem 10.3: Step response of a servo using an ideal relay

A servo using an ideal relay has only second order dynamics as shown in Fig.1. Describe and explain the expected step response.

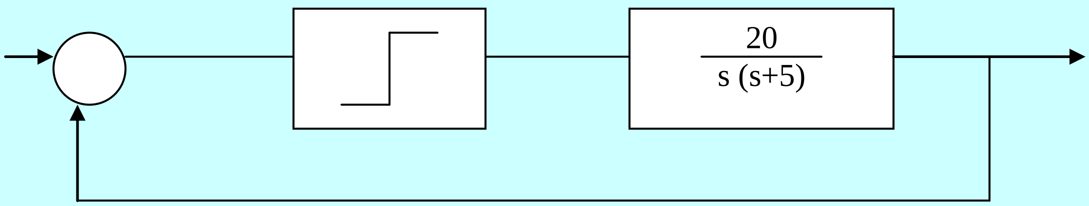

Fig.1

Solution:

For a second order servo with ideal relay there cannot be a limit cycle. Root locus analysis shows that the roots can never cross into the RHP. However, as the transient oscillations decrease in amplitude the relay 'gain' increases moving the roots to higher frequency. Thus in the response to a step input there will be an initial overshoot, but the amplitude of subsequent oscillations decreases, approaching zero, the frequency increases approaching infinity. Sketch a typical response.

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Problem 10.4: Describing function of a nonlinear amplifier

(a) The input x (t) and the output y (t) of a nonlinear device are related through the differential equation

$$y(t) = (dx/dt)^3 + x^2 (dx/dt).$$

Determine the describing function for this device.

(b) Fig. 2 shows the gain characteristic of a nonlinear amplifier.

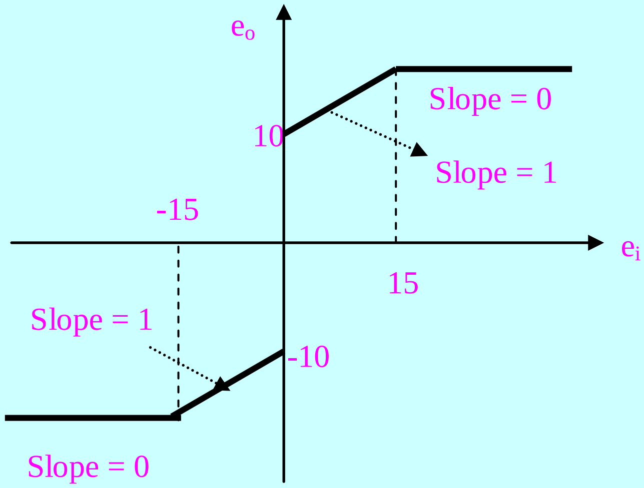

Fig.2

Determine the describing function of the amplifier

<u>Solution:</u>

Q.6 (a) Let x= A sin wt

Then dx/dt = Aw coswt

And $y(t) = (Aw\ coswt)^{3} + (Asinwt)^{2} Awcoswt$

$=0.25A^3w^3\ (cos3wt+3coswt) + A^3wcoswt - 0.25A^3w\ (cos3wt-3coswt)$

Hence, $N = (0.75\ A^2w^3 + 0.25A^2w)\ e^{j\pi/2}$

Q.6 (b)

Let $x = A\ Sin\theta$ where $A > 15$ and let $\alpha = \sin^{-1}(15/A)$

$$\text{Then, } B1 = (4/\pi)\ [\ \int_0^{\alpha}(10 + A\sin\theta)\sin\theta d\theta\ + \int_{\alpha}^{\pi/2} 25\sin\theta d\theta$$

$= (0.25\pi)\ [0.5A\ \sin^{-1}(15/A) + (7.5/A)\ sqrt\ (A^2-225)$

Hence N (A) = B1/A

$= (2/\pi)\ [\sin^{-1}(15/A) + (15/A^2)\ \mathrm{sqrt}\ (A^2-225)$

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Problem 10.5: Describing function of a nonlinear spring

Figs 3(a) and 3(b) show respectively the characteristic curves for a nonlinear spring and that for the nonlinear gain of an amplifier. Derive their respective describing functions.

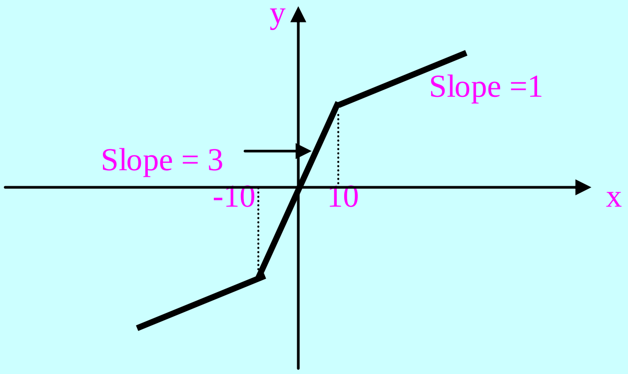


Fig.3 (a)

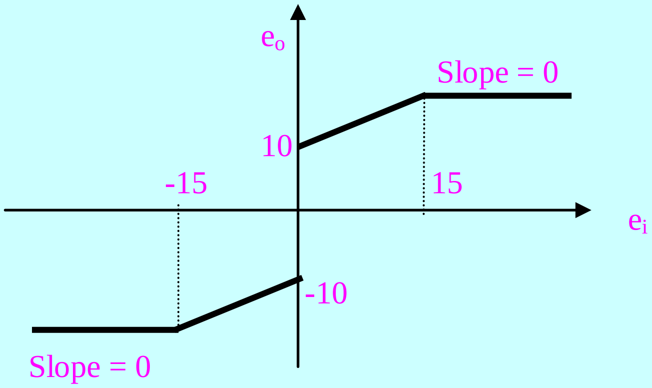
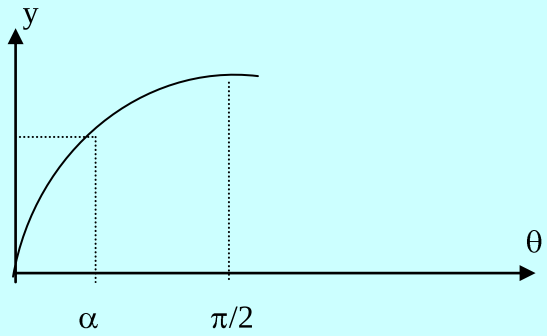


Fig.3 (b)

Solution: **Nonlinear spring:**

Let $x = A \sin\theta$, where $A > 10$
Let $\alpha = \sin^{-1}(10/A)$



Then

$$B1 = (4/\pi)\left[\int_0^{\alpha} 3A \sin^2\theta d\theta + \int_{\alpha}^{\pi/2} \{30 + (A. \sin\theta - 10)\} \sin\theta d\theta\right]$$

$$= (12A / \pi) \int_0^{\alpha} \sin^2\theta d\theta + (80/ \pi) + \int_{\alpha}^{\pi/2} \sin\theta d\theta + (4A/ \pi) \int_{\alpha}^{\pi/2} \sin^2\theta d\theta$$

$$= (6A \alpha/ \pi) - (3A/ \pi) \sin 2\alpha + (80/ \pi) \cos\alpha + A - 2A\alpha/ \pi + (A/ \pi) \sin 2\alpha$$

$$= (4A/\pi) \sin^{-1}(10/A) - (2A/\pi) .2. (1/A) \text{ sqrt } (1-(100/A^2)) + (80/\pi) \text{ sqrt } (1-(100/A^2)) + A$$

Hence,

$$N(A) = 1 + (4/A) \sin^{-1}(10/A) + (76/\pi). (1/A) \text{ sqrt } (1-(100/A^2))$$

Nonlinear amplifier

Let $x = A \sin\theta$, where $A > 15$

Let $\alpha = \sin^{-1}(15/A)$

Then

$$B1 = (4/\pi)\left[\int_0^{\alpha} (10 + A\sin\theta) d\theta + \int_{\alpha}^{\pi/2} 25 \sin\theta d\theta\right]$$

$$= (4/ \pi) [-10\cos\alpha + A \alpha/ 2 - (A/ 4) \sin 2\alpha + 25\cos\alpha]$$

$$= (4/\pi)[(A/2) \sin^{-1}(15/A) + (7.5/A) \text{ sqrt } (A^2-225)]$$

Hence,

N (A) = B1/A== $(2/\pi)$ [$\sin^{-1}$ (15/A) + $(15/A^2)$ sqrt $(A^2$-225)]

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Problem 10.6: Describing function of an on-off nonlinearity

(a) State the available techniques for analyzing non-linear systems
(b) What is the justification of neglecting harmonics in the describing function technique?
(c) Find the describing function of the on-off nonlinearity shown in Fig.4 when the input is E $\sin\omega t$.

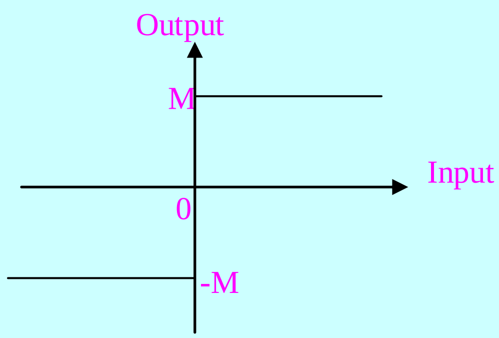


Fig.4

Solution:

(c) Describing function $=4M/\pi E$

TOP

Problem 10.7: Phase plane trajectory of a nonlinear differential equation

(a) State the properties of linear systems which are not valid for non-linear systems
(b) Draw the phase-plane trajectory corresponding to the differential equation

$d^2x/dt^2 + dx/dt + x^3 = 0$

with $x(0) = 1$, $dx/dt(0) = 0$

Solution:

(b)

The equation does not contain a term in x with a positive term. Choose w =1 and write the eqn. as

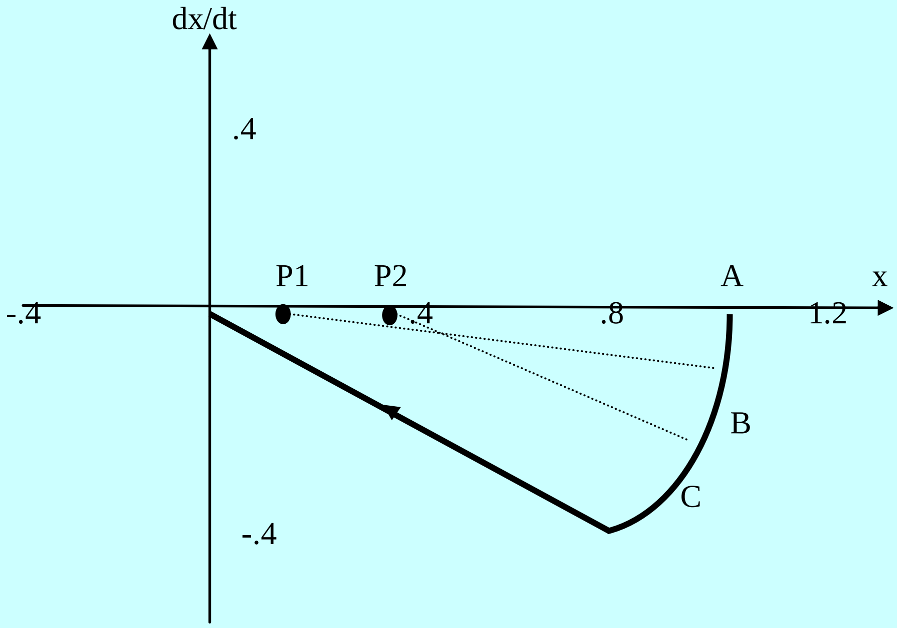
$d^2x/dt^2 + x = -dx/dt - x^3 + x$

$\delta = -dx/dt - x^3 + x$

The trajectory starts at point x=1, dx/dt =0 A short arc is drawn. The mean of x and of dx/dt are used for finding a more

accurate value of δ . The first arc of AB is centered at point P1 (x=.12, dx/dt= 0). BC is centered at P2 (x=.37, dx/dt= 0).

Full trajectory is drawn below.



[TOP]

Problem 10.8: Stability of a nonlinear device using Liapunov's method

(a) Verify that

$V = x_1^4 + 2x_2^2 + 2x_1x_2 + x_1^2$

is a Liapunov function for the system described by the equations

$dx_1/dt = x_2$
$dx_2/dt = -x_2 - x_1^3$

(b) The differential equation of a nonlinear device is

$d^2x/dt^2 + 2x^2 .dx/dt + x = 0$ Use Liapunov's method to determine its stability.

Solution:

(a)

$V= x1^4+x2^2+(x1+x2)^2 \Longrightarrow$ positive definite

$dV/dt = 4x1^3 dx1.dt+4x2dx2/dt+2dx1/dt.x2+2dx2/dt.x1+2x1.dx1/dt$

$= 4x1^3 x2+4x2(-x2-x1^3) +2x2^2+2x1(-x2-x1^3) +2x1x2$

$= -2x2^2-2x1^4 \Longrightarrow$ negative definite.

Hence V is a Liapunov function showing that the system is stable.

(b)

Let x1 = x, x2 = dx/dt

dx1/dt = x2

$dx2/dt = -2x1^2 x2-x1$

If $V= x1^2+x2^2$, then

$dV/dt =2x1.dx1/dt+2x2.dx2/dt=-4x1^2.x2^2 \Longrightarrow$ negative semi-definite

Hence, system is stable.

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Problem 10.9: Stability of nonlinear systems and limit cycles

(a) Determine the describing function for the nonlinear amplifier. Fig. 5 shows its characteristic.

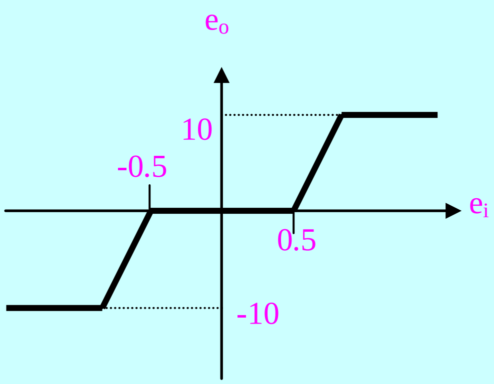


Fig.5

(b) Fig. 6 shows a position control system. The transfer function of the servomotor is

$$G(s) = K\, e^{-0.1s}/[s(s+10)]$$

It is driven by a nonlinear amplifier. Fig. 5 of Part (a) shows the amplifier characteristic

i. Investigate the stability of the system.
ii. What is the maximum value of K for no limit cycle to exist?-

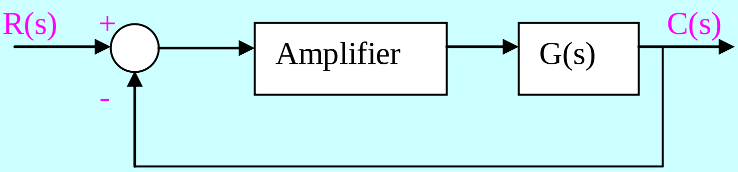


Fig. 6

Solution:

(a)

Consider x= A .sinθ, where A>1

Let $\alpha = \sin^{-1}(.5/A)$, $\beta = \sin^{-1}(1/A)$,

Then, $B1 = (4/\pi)\,[\int_{\alpha}^{\beta} 20[(A\sin\theta - 0.5)\,d\theta \int_{\alpha}^{\pi/2} 10\sin\theta\,d\theta]$

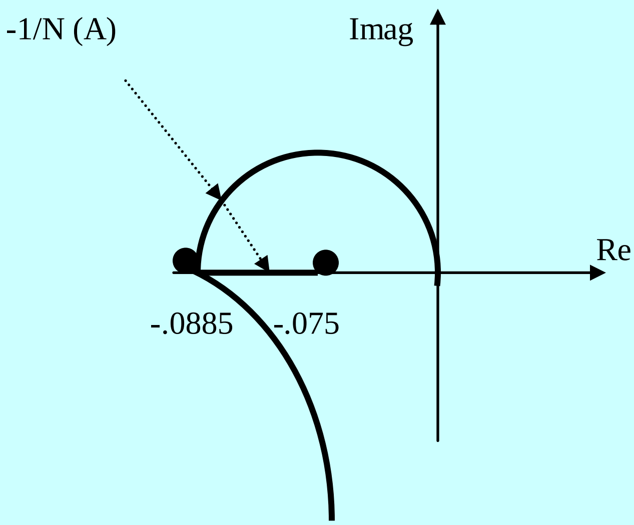
$= (4/\pi)\,[10A(\beta-\alpha) + 10(\cos\beta - \cos\alpha) + 10\cos\alpha$

$= (40A/\pi)\,(\sin^{-1}(1/A) - \sin^{-1}(0.5/A)) + 40.\mathrm{sqrt}\,(A^2-1)/(\pi.A)$

Hence, $N(A) = (40A/\pi)\,(\sin^{-1}(1/A) - \sin^{-1}(0.5/A) + \mathrm{sqrt}\,(A^2-1)/(A^2)$

(b)

For $G(s) = 10e^{-0.1s}/(s(s+1))$, we obtain the frequency response.



The system will be unstable for K=10.

For the system to be stable, the gain must be less than 10*.075/.0885 i.e. K<8.52

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Problem 10.10: To linearize a nonlinear differential equation

1 (a) A fluid reservoir system has a tank of cross-sectional area A in which the fluid depth is H (t). The input and output flow

rates (i.e., fluid volume per unit time) are X (t) and Y (t), so that

$$X(t) - Y(t) = A \cdot \frac{dH(t)}{dt}$$

The nature of the outlet valve is such that

$$Y(t) = C\sqrt{H(t)}$$

with C being a constant. Obtain a nonlinear differential equation relating X and Y, not containing H.

(b) For the system in Part (a), take

$$X(t) = \bar{X} + x(t)$$

$$H(t) = \bar{H} + h(t), \text{ etc}$$

And obtain a linear incremental model relating x (t) and y (t) in terms of $\bar{H}$. Also state the relationships between $\bar{X}$, $\bar{H}$, and $\bar{Y}$.

Hint: first linearize the valve equation.

Solution:

(a) $Y = CH^{1/2} \Rightarrow H = Y^2/C^2 \Rightarrow dH/dt = 2Y.dY/dt/C^2$

$$X - Y = A.dH/dt = \frac{2AY.dY/dt}{C^2}$$

Therefore, $\dfrac{2AY.dY/dt}{C^2} + Y = X$

(b) $Y = CH^{1/2} \Rightarrow$

$$\bar{Y} + y = C(\bar{H} + h)^{1/2} \sim C.\text{sqrt}(\bar{H})(1 + h/2\bar{H})$$

$$\bar{Y} = C.\text{sqrt}(\bar{H}),\ y = Ch/2\text{sqrt}(H)$$

$X - Y = AdH/dt \Rightarrow \quad \bar{X} + x - \bar{Y} - y + A(d\bar{H}/dt + d\bar{h}/dt) = Adh/dt$

Therefore, $\overline{X} = \overline{Y}$, x-y= Adh/dt = $\dfrac{2\text{sqrt}(\overline{H}).A.dy/dt}{C}$

$\dfrac{2\text{sqrt}(\overline{H}).A.dy/dt}{C} + y = x$

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Problem 10.11: Stability of nonlinear systems and limit cycles

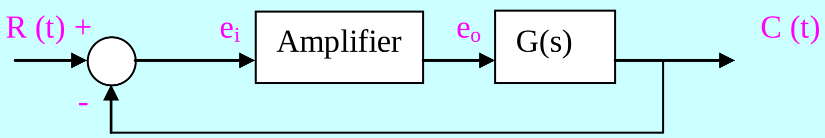


Fig.8

The transfer function

$G(s) = K\, e^{-0.1s}[s(s+10)]$

of the system in Fig.8 has a gain of 20 in the linear range, and it saturates when its input exceeds 1.

i. Investigate the stability of the system when K= 10
ii. What is the maximum value of K for no limit cycle to exist?

Solution:

(a)

Let x= A sin wt
Then dx/dt= Aw coswt
Y (t) = $(Aw\cos wt)^3 + (A\sin wt)^{2Aw} \cos wt = (1/4) A^3 (\cos 3wt+3\cos wt) + A^3 w \cos wt - (1/4).\ A^3 w (\cos 3wt-3\cos wt)$.
Hence, N= $(0.75\, A^2 w^3 + .25\, A^2) \exp(j\pi/2)$

(b)

Let x= A sin wt, A>1 and $\alpha = \sin^{-1}(1/A)$

$$\text{Then } B1 = (4/\pi)\left[\int_0^{\alpha} 20A\sin^2\theta\, d\theta + \int_{\alpha}^{\pi/2} 20\sin\theta\, d\theta\right]$$

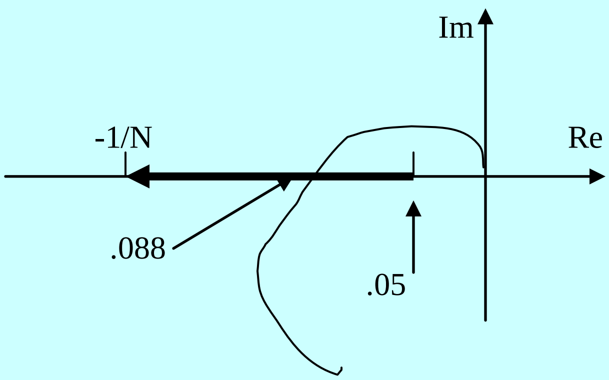
$$= (80/\pi)\left[(A/2)\sin^{-1}(1/A) - (1/2)\,\mathrm{sqrt}\,((1-(1/A^2)) + \mathrm{sqrt}\,((1-(1/A^2))\right]$$

$$N(A) = (40/\pi)\left[\sin^{-1}(1/A) + (1/A)\,\mathrm{sqrt}\,((1-(1/A^2))\right)$$

For K=10, G(s) = 10 exp (-0.1s)/(s(s+10))

The following frequency response is obtained:

W	1	2	4	6	8.603	1
Phase (deg.)	-101.4	-112.7	-134.7	-155.3	-180	-192.3
M	.995	.49	.232	.143	.088	.071



a. The system will be unstable for K=10
b. For the system to be stable, gain must be < 10* 0.05/.088

i.e., K<5.682

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Problem 10.12: Stability of nonlinear systems and limit cycles

For the block diagram in Fig.9,

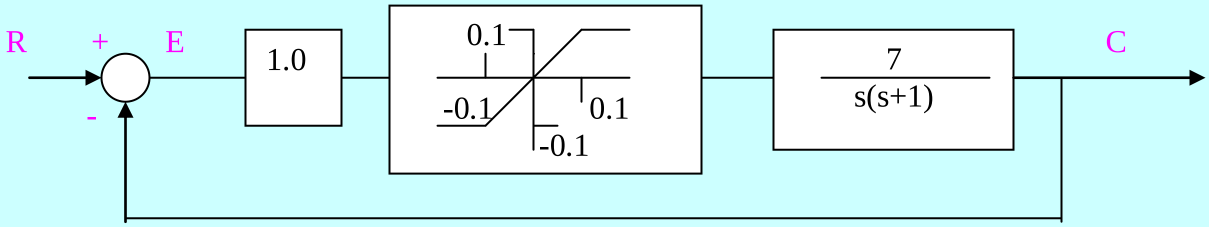

Fig.9

a) Derive the applicable differential equations.
b) On the E versus dE/dt plane, where, if any, are the singular points?
c) What are the equations of the dividing lines?
d) What are the equations of the isoclines?
e) Obtain several trajectories on the E versus dE/dt plane. Show singular points, dividing lines, and some isoclines.

Solution:

$$C/E = \frac{7}{s(s+1)}$$

$\Longrightarrow$ $s^2C + sC = 7E$ $(-.1<E<.1)$

$d^2C/dt^2 = dC/dt = 7E$

Bur for R= constant, $dc/dt = -\,dE/dt$

$d^2C/dt^2 = -\,d^2E/dt^2$

Therefore, $d^2E/dt^2 + dE/dt + 7E = 0$

When $E>.1$,

$d^2E/dt^2 + dE/dt + .7 = 0$

When $E< .1$,

$d^2E/dt^2 + dE/dt - .7 = 0$

(b) For R=0 or R= constant there is a singular point at $E=0, dE/dt=0$

(c) The dividing line equations are E = .1

E= -.1

(d) In the linear zone (-.1<E<.1)

d^2E/dt^2 =- dE/dt - 7E

d^2E/dt^2 /(dE/dt) = - dE/dt/(dE/dt) - 7E/(dE/dt)

d^2E/dt^2 /(dE/dt) ={ **-ddE/dt /(dE/dt) = ddE/dt /(dE/dt) = N (by definition)}** = -dE/dt –7E

(N+1) dE/dt=-7E

dE/dt = -7E/ (N+1) Isocline equation

When E> .1 dE/dt= -.7/ (N+1)
When E< -.1, dE/dt= .7/ (N+!)

The dividing lines, singular points and isoclines are to be shown on a sketch.

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Objective type questions:

(i) The describing function of the nonlinear input- output characteristic, of a device, shown in Fig.3

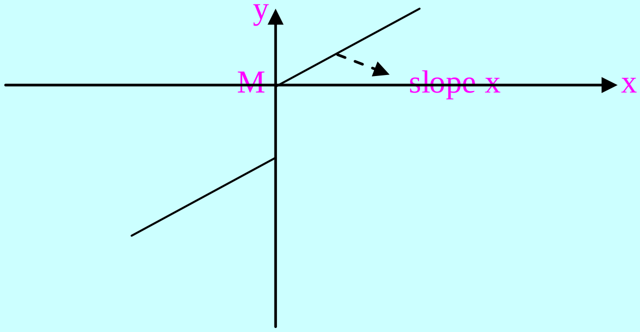


Fig.3

is obtained as

(a) $M + (4k/\pi X)$
(b) $X + (4M/\pi k)$
(c) $k + (4M/\pi X)$

where X is the amplitude of the input sinusoid.

Ans: (c)

(ii) Which one of the following statements is <u>not</u> true?

(a) Principal of superposition is not applicable to nonlinear systems.
(b) Altering the size of the input does not change the response of a linear system.
(c) Altering the size of the input affects the percentage overshoot of the response of a nonlinear system but the frequency of oscillation is independent of the size of the input.
(d) In nonlinear systems, stability may depend on the magnitude of the input as well as on the initial conditions

Ans: (c)

(iii) A nonlinear amplifier is described by the following characteristic:

$$v_o(t) = \begin{cases} v_{in}^2 & v_{in} \geq 0 \\ -v_{in}^2 & v_{in} < 0 \end{cases}$$

The amplifier is operated over a range for v_{in} of $\pm$ 0.5 volts at the operating point. The amplifier can be described by the following linear approximation when the operating point is 1 volt.

(a) $v_o(t) = 2v_{in}$
(b) $v_o(t) = v_{in} / 2$
(c) $v_o(t) = 2v_{in} - 1$

Ans: (c)

(iv) The eigenvector of the matrix

$$\begin{pmatrix} -2 & -5 & -5 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

corresponding to an eigenvalue $\lambda = -1$ is

(a) $[0 \quad -1 \quad 1]^T$

(b) $[-3-j4, -1+j2, 1]^T$

(c) $[-3+j4, -1-j2, 1]^T$

[note that T is the transpose of a vector]

Ans: (a)

(v) The value of 6 dB is equal to

(a) 12dB per decade
(b) –6 dB per decade
(c) 20 dB per decade

Ans: (c)

(vi) Which one of the following statements is true?

(a) For studying the stability of continuous systems, Liapunov’s method is applicable for linear systems only

(b)For studying the stability of continuous systems, Liapunov’s method is applicable for nonlinear systems only

(c)For studying the stability of continuous systems, Liapunov’s method is applicable for both linear and nonlinear systems

Ans: (c)

(vii) Gear trains in control systems are a cause of the following nonlinearity.

(a) saturation
(b) dead-space
(c) coulomb friction
(d) backlash

Ans: (b)

(viii) Which one of the following statements is <u>not</u> true?

(a) A limit cycle in a nonlinear system describes the amplitude and period of a self-sustained oscillation.

(b) A limit cycle is stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle.

(c) A limit cycle is unstable if trajectories near it diverge from the limit cycle. In this case, an unstable region surrounds a stable region.

(d) A limit cycle is unstable if trajectories near it diverge from the limit cycle. In this case, if a trajectory starts with the stable region, it converges to singular point within the limit cycle. If a trajectory starts in the unstable region, it diverges and increases with time to infinity.

Ans: (c)

(ix) Which one of the following statements is true?

a) The Liapunov method is the most general method for determining the controllability of nonlinear and/or time-varying systems.
b) The Liapunov method is the most general method for determining the observability of nonlinear and/or time-varying systems.
c) The Liapunov method is the most general method for determining the stability of nonlinear and/or time-varying systems.

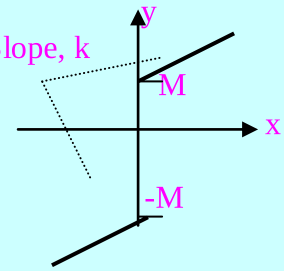
Ans: (a)

(x) Match List E with List F in the following Table of nonlinearities and their describing functions, N

List E- List F

A-

Slope, k

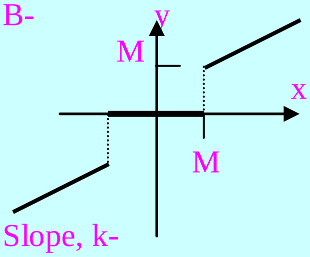


I-

$$N = k + (4M/\pi x)$$

-I-

B-



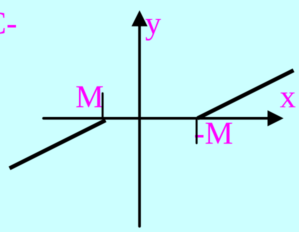
Slope, k-

II-

$$N = k - (2k/\pi)\sin^{-1}(M/x) + \frac{(4-2k)\,M.\,\mathrm{sqrt}\,(1-(M^2/x^2))}{\pi x}$$

$x >$ or $= M$

C-



slope, k

-III-

$$N = k - (2k/\pi)\,[\sin^{-1}(M/x) + (M/x)\,.\mathrm{sqrt}\,(1-(M2/x^2))]$$

The correct matching is

(a) AIII BII C I
(b) AI B III C II
(c) A II B III C I
(d) A I B II C III

Ans: (d)

(xi) Which one of the following is <u>not</u> a property of the Liapunov function?

a) It is a unique function for a given system.
b) Its time derivative is non-positive.
c) It is positive definite, at least in the neighbourhood of the origin.

Ans: a

(xii) Which one of the following statements is <u>not</u> true?

(a)The describing function analysis is an extension of linear techniques applied to nonlinear systems.
(b)Typical applications of describing function analysis are to systems with a <u>low</u> degree of nonlinearity.
(c) Typical applications of describing function analysis are to systems with a <u>high</u> degree of nonlinearity.
(d)The describing function method is an approximate method for determining the stability of unforced control systems.

Ans: ©

(xiii)Match List E with List F in the following Table of common nonlinearities and their characteristics.

Question

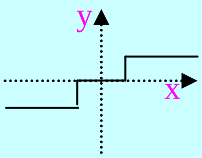
List E

ListF

A

Saturation

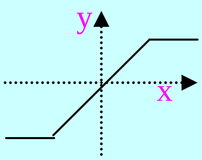
I



B

Ideal relay

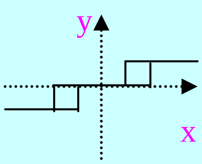
II



C

Relay with a dead zone

III

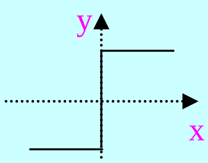


D

Relay with a dead

zone and hysterisis

IV



The correct matching is

(a) AIII BII CIV DI
(b) AIV BI CII DIII
(c) AII BIV CI DIII
(d) AI BIVCIII DII

Ans: ©

(xiv) Which one of the following statements is true?

(a) The Liapunov method is the most general method for determining the <u>controllability</u> of nonlinear and/or time-varying systems.

(b) The Liapunov method is the most general method for determining the <u>observability</u> of nonlinear and/or time-varying systems.

(c) The Liapunov method is the most general method for determining the <u>stability</u> of nonlinear and/or time-varying systems.

Ans: c

(xv) List E gives several scalar functions and list F gives their classifications. Match list E with List F.

List E-	List F-
A-$V(x) = x_1^2 + 2x_2^2$-	I-Indefinite
B-$V(x) = (x_1+x_2)^2$-	II-Negative definite
C-$V(x) = -x_1^2-(3x_1+2x_2)^2$-	III-Positive semi-definite
D-$V(x) = x_1x_2+x_2^2$-	IV-Positive definite

The correct matching is

(a) AIV, BIII, CII, DI
(b) AI, BII, CIII, DIV
(c) AIV, BIII, CI, DII

Ans: a

(xvi) A nonlinear system defined by dx/dt = f(x) is asymptotically stable in the vicinity of the equilibrium point at the origin if there exists a scalar function V such that

(a) V (x) is discontinuous and has no continuous first partial derivatives at the origin
(b) V (x) = 0 for x not equal to 0
© V (x)< 0 for all x not equal to 0

Ans: (c)
TOP